Poisson boundary of the discrete quantum group $\widehat{A_u(F)}$

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Abstract

We identify the Poisson boundary of the dual of the universal compact quantum group $A_u(F)$ with a measurable field of ITPFI factors.

1 Introduction and statement of main result

Poisson boundaries of discrete quantum groups were introduced by Izumi [5] in his study of infinite tensor product actions of $\mathrm{SU}_q(2)$. Izumi was able to identify the Poisson boundary of the dual of $\mathrm{SU}_q(2)$ with the quantum homogeneous space $\mathrm{L}^\infty(\mathrm{SU}_q(2)/S^1)$, called the Podleś sphere. The generalization to $\mathrm{SU}_q(n)$ was established by Izumi, Neshveyev and Tuset [6], yielding $\mathrm{L}^\infty(\mathrm{SU}_q(n)/S^{n-1})$ as the Poisson boundary. A more systematic approach was given by Tomatsu [10] who proved the following very general result: if $\mathbb{G}$ is a compact quantum group with commutative fusion rules and amenable dual $\widehat{\mathbb{G}}$, the Poisson boundary of $\widehat{\mathbb{G}}$ can be identified with the quantum homogeneous space $\mathrm{L}^\infty(\mathbb{G}/\mathbb{K})$, where $\mathbb{K}$ is the maximal closed quantum subgroup of Kac type inside $\mathbb{G}$. Tomatsu's result provides the Poisson boundary for the duals of all q-deformations of classical compact groups.

All examples discussed in the previous paragraph concern amenable discrete quantum groups. In [12], we identified the Poisson boundary for the (non-amenable) dual of the compact quantum group $A_o(F)$ with a higher dimensional Podleś sphere. Although the dual of $A_o(F)$ is non-amenable, the representation category of $A_o(F)$ is monoidally equivalent with the representation category of $\mathrm{SU}_q(2)$ for the appropriate value of q. The second author and De Rijdt provided in [4] a general result explaining the behavior of the Poisson boundary under the passage to monoidally equivalent quantum groups. In particular, a combination of the results of [4] and [5] give a more conceptual approach to our identification in [12].

The quantum random walks studied on a discrete quantum group $\widehat{\mathbb{G}}$ have a semi-classical counterpart, being a Markov chain on the (countable) set $\mathrm{Irred}(\mathbb{G})$ of irreducible representations of $\mathbb{G}$ (modulo unitary equivalence). All the examples above, share the feature that the semi-classical random walk on $\mathrm{Irred}(\mathbb{G})$ has trivial Poisson boundary.

In this paper, we identify the Poisson boundary for the dual of $\mathbb{G} = A_u(F)$. In that case, $\mathrm{Irred}(\mathbb{G})$ can be identified with the Cayley tree of the monoid $\mathbb{N} * \mathbb{N}$ and, by results of [9], has a non-trivial Poisson boundary: the end compactification of the tree with the appropriate harmonic measure. Before discussing in more detail our main result, we introduce some terminology and notations. For a more complete introduction to Poisson boundaries of discrete quantum groups, we refer to [15, Chapter 4].

Compact quantum groups were originally introduced by Woronowicz in [17] and their definition finally took the following form.

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Definition 1.1 (Woronowicz [18, Definition 1.1])**.** A *compact quantum group* $\mathbb{G}$ is a pair consisting of a unital C*-algebra $\mathrm{C}(\mathbb{G})$ and a unital $*$-homomorphism $\Delta : \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$, called *comultiplication,* satisfying the following two conditions.

- *Co-associativity:* $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
- $\operatorname{span} \Delta(\mathrm{C}(\mathbb{G}))(1 \otimes \mathrm{C}(\mathbb{G}))$ and $\operatorname{span} \Delta(\mathrm{C}(\mathbb{G}))(\mathrm{C}(\mathbb{G}) \otimes 1)$ are dense in $\mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$.

In the above definition, the symbol $\otimes$ denotes the minimal (i.e. spatial) tensor product of C*-algebras.

Let $\mathbb{G}$ be a compact quantum group. By [18, Theorem 1.3], there is a unique state h on $\mathrm{C}(\mathbb{G})$ satisfying $(\mathrm{id} \otimes h)\Delta(a) = h(a)1 = (h \otimes \mathrm{id})\Delta(a)$ for all $a \in \mathrm{C}(\mathbb{G})$. We call h the *Haar state* of $\mathbb{G}$.

A *unitary representation* of $\mathbb{G}$ on the finite dimensional Hilbert space H is a unitary operator $U \in \mathcal{L}(H) \otimes \mathrm{C}(\mathbb{G})$ satisfying $(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}$. Given unitary representations U_1, U_2 on H_1, H_2, we put

$$\mathrm{Mor}(U_2, U_1) := \{S \in \mathcal{L}(H_1, H_2) \mid (S \otimes 1)U_1 = U_2(S \otimes 1)\} .$$

Let U be a unitary representation of $\mathbb{G}$ on the finite dimensional Hilbert space H. The elements $(\xi^* \otimes 1)U(\eta \otimes 1) \in \mathrm{C}(\mathbb{G})$ are called the coefficients of U. The linear span of all coefficients of all finite dimensional unitary representations of $\mathbb{G}$ forms a dense $*$-subalgebra of $\mathrm{C}(\mathbb{G})$ (see [18, Theorem 1.2]). We call U *irreducible* if $\mathrm{Mor}(U, U) = \mathbb{C}1$. We call U_1 and U_2 *unitarily equivalent* if $\mathrm{Mor}(U_2, U_1)$ contains a unitary operator.

Let U be an irreducible unitary representation of $\mathbb{G}$ on the finite dimensional Hilbert space H. By [18, Proposition 5.2], there exists an anti-linear invertible map $j : H \to \overline{H}$ such that the operator $U^c \in \mathcal{L}(\overline{H}) \otimes \mathrm{C}(\mathbb{G})$ defined by the formula $(j(\xi)^* \otimes 1)U^c(j(\eta) \otimes 1) = (\eta^* \otimes 1)U^*(\xi \otimes 1)$ is unitary. One calls U^c the *contragredient* of U. Since U is irreducible, the map j is uniquely determined up to multiplication by a non-zero scalar. We normalize in such a way that $Q := j^*j$ satisfies $\mathrm{Tr}(Q) = \mathrm{Tr}(Q^{-1})$. Then, j is determined up to multiplication by $\lambda \in S^1$ and Q is uniquely determined. We call $\mathrm{Tr}(Q)$ the *quantum dimension* of U and denote it by $\dim_q(U)$. Note that $\dim_q(U) \geq \dim(H)$ with equality holding iff $Q = 1$.

The *tensor product* $U \mathbin{\textcircled{\scriptsize T}} V$ of two unitary representations is defined as $U_{13}V_{23}$.

Given a compact quantum group $\mathbb{G}$, we denote by $\mathrm{Irred}(\mathbb{G})$ the set of irreducible unitary representations of $\mathbb{G}$ modulo unitary conjugacy. For every $x \in \mathrm{Irred}(\mathbb{G})$, we choose a representative U^x on the Hilbert space H_x. We denote by $Q_x \in \mathcal{L}(H_x)$ the associated positive invertible operator and define the state ψ_x on $\mathcal{L}(H_x)$ by the formula

$$\psi_x(A) := \frac{\mathrm{Tr}(Q_x A)}{\mathrm{Tr}(Q_x)} .$$

The dual, *discrete quantum group* $\widehat{\mathbb{G}}$ is defined as the ℓ^∞-direct sum of matrix algebras

$$\ell^\infty(\widehat{\mathbb{G}}) := \prod_{x \in \mathrm{Irred}(\mathbb{G})} \mathcal{L}(H_x) .$$

We denote by $p_x, x \in \mathrm{Irred}(\mathbb{G})$, the minimal central projections in $\ell^\infty(\widehat{\mathbb{G}})$. Denote by $\epsilon \in \mathrm{Irred}(\mathbb{G})$ the trivial representation and by $\widehat{\epsilon} : \ell^\infty(\widehat{\mathbb{G}}) \to \mathbb{C}$ the co-unit given by $ap_\epsilon = \widehat{\epsilon}(a)p_\epsilon$.

Whenever $x, y, z \in I$, we use the short-hand notation $\mathrm{Mor}(x \otimes y, z) := \mathrm{Mor}(U^x \mathbin{\textcircled{\scriptsize T}} U^y, U^z)$ and we write $z \subset x \otimes y$ if $\mathrm{Mor}(x \otimes y, z) \neq \{0\}$.

The von Neumann algebra $\ell^\infty(\widehat{\mathbb{G}})$ carries a comultiplication $\hat{\Delta} : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\widehat{\mathbb{G}})$, uniquely characterized by the formula

$$\hat{\Delta}(a)(p_x \otimes p_y)S = Sap_z \quad \text{for all} \quad x, y, z \in \mathrm{Irred}(\mathbb{G}) \;\text{ and }\; S \in \mathrm{Mor}(x \otimes y, z) \; .$$

Denote by $\mathrm{L}^\infty(\mathbb{G})$ the weak closure of $\mathrm{C}(\mathbb{G})$ in the GNS representation of the Haar state h. One defines the unitary $\mathbb{V} \in \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{G})$ by the formula

$$\mathbb{V} := \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} U^x \; .$$

The unitary $\mathbb{V}$ implements the duality between $\mathbb{G}$ and $\widehat{\mathbb{G}}$, in the sense that it satisfies

$$(\hat{\Delta} \otimes \mathrm{id})(\mathbb{V}) = \mathbb{V}_{13}\mathbb{V}_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta)(\mathbb{V}) = \mathbb{V}_{12}\mathbb{V}_{13} \; .$$

Discrete quantum groups can also be defined intrinsically, see [13].

Whenever $\omega \in \ell^\infty(\widehat{\mathbb{G}})_*$ is a normal state, we consider the Markov operator

$$P_\omega : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{G}}) : P_\omega(a) = (\mathrm{id} \otimes \omega)\hat{\Delta}(a) \; .$$

By [7, Proposition 2.1], the Markov operator P_ω leaves globally invariant the center $\mathcal{Z}(\ell^\infty(\widehat{\mathbb{G}}))$ of $\ell^\infty(\widehat{\mathbb{G}})$ if and only if

$$\omega = \psi_\mu := \sum_{x \in \mathrm{Irred}(\mathbb{G})} \mu(x)\psi_x \quad \text{where} \quad \mu \;\text{ is a probability measure on }\; \mathrm{Irred}(\mathbb{G}) \; .$$

We only consider states ω of the form ψ_μ and denote by P_μ the corresponding Markov operator. Note that we can define a convolution product on the probability measures on $\mathrm{Irred}(\mathbb{G})$ by the formula

$$P_{\mu * \eta} = P_\mu \circ P_\eta \; .$$

Considering the restriction of P_μ to $\ell^\infty(\mathrm{Irred}(\widehat{\mathbb{G}})) = \mathcal{Z}(\ell^\infty(\widehat{\mathbb{G}}))$, every probability measure μ on $\mathrm{Irred}(\mathbb{G})$ defines a Markov chain on the countable set $\mathrm{Irred}(\mathbb{G})$ with n-step transition probabilities given by

$$p_x p_n(x, y) = p_x P_\mu^n(p_y) \; .$$

The probability measure μ is called *generating* if for every $x, y \in \mathrm{Irred}(\mathbb{G})$, there exists an $n \in \mathbb{N} \backslash \{0\}$ such that $p_n(x, y) > 0$.

Definition 1.2. Let $\mathbb{G}$ be a compact quantum group and μ a generating probability measure on $\mathrm{Irred}(\mathbb{G})$. The *Poisson boundary* of $\widehat{\mathbb{G}}$ with respect to μ is defined as the space of P_μ-harmonic elements in $\ell^\infty(\widehat{\mathbb{G}})$.

$$\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu) := \{a \in \ell^\infty(\widehat{\mathbb{G}}) \mid P_\mu(a) = a\} \; .$$

The weakly closed vector subspace $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$ of $\ell^\infty(\widehat{\mathbb{G}})$ is turned into a von Neumann algebra using the product (cf. [5, Theorem 3.6])

$$a \cdot b := \lim_{n \to \infty} P_\mu^n(ab)$$

and where the sequence at the right hand side is strongly* convergent.

- The restriction of $\hat{\epsilon}$ to $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$ is a faithful normal state on $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$.

- The restriction of $\hat{\Delta}$ to $\mathrm{H}^\infty(\widehat{\mathbb{G}},\mu)$ defines a left action

$$\alpha_{\widehat{\mathbb{G}}} : \mathrm{H}^\infty(\widehat{\mathbb{G}},\mu) \to \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} \mathrm{H}^\infty(\widehat{\mathbb{G}},\mu) : a \mapsto \hat{\Delta}(a)$$

  of $\widehat{\mathbb{G}}$ on $\mathrm{H}^\infty(\widehat{\mathbb{G}},\mu)$.

- The restriction of the adjoint action to $\mathrm{H}^\infty(\widehat{\mathbb{G}},\mu)$ defines an action

$$\alpha_{\mathbb{G}} : \mathrm{H}^\infty(\widehat{\mathbb{G}},\mu) \to \mathrm{H}^\infty(\widehat{\mathbb{G}},\mu) \overline{\otimes} \mathrm{L}^\infty(\mathbb{G}) : a \mapsto \mathbb{V}(a \otimes 1)\mathbb{V} \; .$$

We denote by $\mathrm{H}^\infty_{\mathrm{centr}}(\widehat{\mathbb{G}},\mu) := \mathrm{H}^\infty(\widehat{\mathbb{G}},\mu) \cap \mathcal{Z}(\ell^\infty(\widehat{\mathbb{G}}))$ the space of bounded P_μ-harmonic functions on $\mathrm{Irred}(\mathbb{G})$. Defining the conditional expectation

$$\mathcal{E} : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\mathrm{Irred}(\widehat{\mathbb{G}})) : \mathcal{E}(a)p_x = \psi_x(a)p_x \; ,$$

we observe that $\mathcal{E}$ also provides a faithful conditional expectation of $\mathrm{H}^\infty(\widehat{\mathbb{G}},\mu)$ onto the von Neumann subalgebra $\mathrm{H}^\infty_{\mathrm{centr}}(\widehat{\mathbb{G}},\mu)$.

We now turn to the concrete family of compact quantum groups studied in this paper and introduced by Van Daele and Wang in [14]. Let $n \in \mathbb{N} \setminus \{0,1\}$ and let $F \in \mathrm{GL}(n,\mathbb{C})$. One defines the compact quantum group $\mathbb{G} = A_u(F)$ such that $\mathrm{C}(\mathbb{G})$ is the universal unital C*-algebra generated by the entries of an $n \times n$ matrix U satisfying the relations

$$U \quad \text{and} \quad F\overline{U}F^{-1} \quad \text{are unitary, with} \quad \left(\overline{U}\right)_{ij} = (U_{ij})^*$$

and such that $\Delta(U_{ij}) = \sum_{k=1}^n U_{ik} \otimes U_{kj}$. By definition, U is an n-dimensional unitary representation of $A_u(F)$, called the *fundamental representation.*

Fix $F \in \mathrm{GL}(n,\mathbb{C})$ and put $\mathbb{G} = A_u(F)$. For reasons to become clear later, we assume that F is not a scalar multiple of a unitary 2×2 matrix.

By [2, Théorème 1], the irreducible unitary representations of $\mathbb{G}$ can be labeled by the elements of the free monoid $I := \mathbb{N} * \mathbb{N}$ generated by α and β. We represent the elements of I as words in α and β. The empty word is denoted by ϵ and corresponds to the trivial representation of $\mathbb{G}$, while α corresponds to the fundamental representation and β to the contragredient of α. We denote by $x \mapsto \overline{x}$ the unique antimultiplicative and involutive map on I satisfying $\overline{\alpha} = \beta$. This involution corresponds to the contragredient on the level of representations. The fusion rules of $\mathbb{G}$ are given by

$$x \otimes y \cong \bigoplus_{z \in I, x = x_0 z, y = \overline{z} y_0} x_0 y_0 \; .$$

So, if the last letter of x equals the first letter of y, the tensor product $x \otimes y$ is irreducible and given by xy. We denote this as $xy = x \otimes y$.

Denote by ∂I the compact space of infinite words in α and β. For $x \in \partial I$, the expression

$$x = x_1 \otimes x_2 \otimes \cdots \tag{1}$$

means that the infinite word x is the concatenation of the finite words $x_1 x_2 \cdots$ and that the last letter of x_n equals the first letter of x_{n+1} for all $n \in \mathbb{N}$. All elements x of ∂I can be decomposed as in (1), except the countable number of elements of the form $x = y\alpha\beta\alpha\beta\cdots$ for some $y \in I$.

Below, we will only deal with non-atomic measures on ∂I, so that almost every point of ∂I has a decomposition as in (1). We denote by $\partial_0 I$ the subset of ∂I consisting of the infinite words that have a decomposition of the form (1).

The following is the main result of the paper.

Theorem 1.3. *Let $F \in \mathrm{GL}(n, \mathbb{C})$ such that F is not a scalar multiple of a unitary 2×2 matrix. Write $\mathbb{G} = A_u(F)$ and suppose that μ is a finitely supported, generating probability measure on $I = \mathrm{Irred}(\mathbb{G})$. Denote by ∂I the compact space of infinite words in the letters α, β. There exists*

- *a non-atomic probability measure ν_ϵ on ∂I,*
- *a measurable field M of ITPFI factors over $(\partial I, \nu_\epsilon)$ with fibers*

$$(M_x, \omega_x) = \bigotimes_{k=1}^{\infty} (\mathcal{L}(H_{x_k}), \psi_{x_k})$$

whenever $x \in \partial_0 I$ is of the form $x = x_1 x_2 x_3 \cdots = x_1 \otimes x_2 \otimes x_3 \otimes \cdots$,

- *an action $\beta_{\widehat{\mathbb{G}}}$ of $\widehat{\mathbb{G}}$ on M concretely given by (2) below,*

such that, with $\omega_\infty = \int^\oplus \omega_x \, d\nu_\epsilon(x)$, the Poisson integral formula

$$\Theta_\mu : M \to \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu) : \Theta_\mu(a) = (\mathrm{id} \otimes \omega_\infty)\beta_{\widehat{\mathbb{G}}}(a)$$

defines a $*$-isomorphism of M onto $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$, intertwining the action $\beta_{\widehat{\mathbb{G}}}$ on M with the action $\alpha_{\widehat{\mathbb{G}}}$ on $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$.

Moreover, defining the action $\beta^x_{\mathbb{G}}$ of $\mathbb{G}$ on M_x as the infinite tensor product of the inner actions $a \mapsto U^{x_k}(a \otimes 1)(U^{x_k})^*$, we obtain the action $\beta_{\mathbb{G}}$ of $\mathbb{G}$ on M. The $*$-isomorphism Θ_μ intertwines $\beta_{\mathbb{G}}$ with $\alpha_{\mathbb{G}}$.

The comultiplication $\hat{\Delta} : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\widehat{\mathbb{G}})$ can be uniquely cut down into completely positive maps $\hat{\Delta}_{x \otimes y, z} : \mathcal{L}(H_z) \to \mathcal{L}(H_x) \otimes \mathcal{L}(H_y)$ in such a way that

$$\hat{\Delta}(a)(p_x \otimes p_y) = \sum_{z \subset x \otimes y} \hat{\Delta}_{x \otimes y, z}(a p_z)$$

for all $a \in \ell^\infty(\widehat{\mathbb{G}})$.

We denote by $|x|$ the length of a word $x \in I$.

If now $x, y \in I, z \in \partial I$ with $yz = y \otimes z$ and $|y| > |x|$, we define for all $s \subset x \otimes y$,

$$\hat{\Delta}_{x \otimes yz, sz} : M_{sz} \to \mathcal{L}(H_x) \otimes M_{yz}$$

by composing $\hat{\Delta}_{x \otimes y, s} \otimes \mathrm{id}$ with the identifications $M_{sz} \cong \mathcal{L}(H_s) \otimes M_z$ and $M_{yz} \cong \mathcal{L}(H_y) \otimes M_z$. The action $\beta_{\widehat{\mathbb{G}}} : M \to \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} M$ of $\widehat{\mathbb{G}}$ on M is now given by

$$\beta_{\widehat{\mathbb{G}}}(a)_{x, yz} = \sum_{s \subset x \otimes y} \hat{\Delta}_{x \otimes yz, sz}(a p_{sz}) \tag{2}$$

whenever $a \in M$, $x, y \in I$, $z \in \partial I$, $|y| > |x|$ and $yz = y \otimes z$. Note that we identified $\ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} M$ with a measurable field over $I \times \partial I$ with fiber in (x, z) given by $\mathcal{L}(H_x) \otimes M_z$.

Further notations and terminology

Fix $F \in \mathrm{GL}(n, \mathbb{C})$ and put $\mathbb{G} = A_u(F)$. We identify $\mathrm{Irred}(\mathbb{G})$ with $I := \mathbb{N} * \mathbb{N}$. We assume that F is not a multiple of a unitary 2×2 matrix. Equivalently, $\dim_q(\alpha) > 2$. The first reason to do so, is that under this assumption, the random walk defined by any non-trivial probability measure μ on I (i.e. $\mu(\epsilon) < 1$), is automatically *transient,* which means that

$$\sum_{n=1}^{\infty} p_n(x,y) < \infty$$

for all $x, y \in I$. This statement can be proven in the same was as [7, Theorem 2.6]. For the convenience of the reader, we give the argument. Denote by $\dim_{\min}(y)$ the dimension of the carrier Hilbert space of y, when y is viewed as an irreducible representation of $A_u(I_2)$. Since F is not a multiple of a unitary 2×2 matrix, it follows that $\dim_q(y) > \dim_{\min}(y)$ for all $y \in I \setminus \{\epsilon\}$. Denote by $\mathrm{mult}(z; y_1 \otimes \cdots \otimes y_n)$ the multiplicity of the irreducible representation $z \in I$ in the tensor product of the irreducible representations $y_1, \ldots, y_n$. Since the fusion rules of $A_u(F)$ and $A_u(I_2)$ are identical, it follows that

$$\mathrm{mult}(z; y_1 \otimes \cdots \otimes y_n) \leq \dim_{\min}(y_1) \cdots \dim_{\min}(y_n) \; .$$

One then computes for all $x, y \in I$, $n \in \mathbb{N}$,

$$\begin{aligned} p_n(x,y) &= \sum_{z \subset \overline{x} \otimes y} \mu^{*n}(z) \frac{\dim_q(y)}{\dim_q(x) \dim_q(z)} \\ &= \frac{\dim_q(y)}{\dim_q(x)} \sum_{z \subset \overline{x} \otimes y} \sum_{y_1, \ldots, y_n \in I} \mathrm{mult}(z; y_1 \otimes \cdots y_n) \frac{\mu(y_1) \cdots \mu(y_n)}{\dim_q(y_1) \cdots \dim_q(y_n)} \\ &\leq \frac{\dim_q(y)}{\dim_q(x)} \dim(\overline{x} \otimes y) \; \rho^n \end{aligned}$$

where $\rho = \displaystyle\sum_{y \in I} \mu(y) \frac{\dim_{\min}(y)}{\dim_q(y)}$. Since μ is non-trivial and F is not a multiple of a 2×2 unitary matrix, we have $0 < \rho < 1$. Transience of the random walk follows immediately.

An element $x \in I$ is said to be *indecomposable* if $x = y \otimes z$ implies $y = \epsilon$ or $z = \epsilon$. Equivalently, x is an alternating product of the letters α and β.

For every $x \in I$, we denote by $\dim_q(x)$ the *quantum dimension* of the irreducible representation labeled by x. Since $\dim_q(\alpha) > 2$, take $0 < q < 1$ such that $\dim_q(\alpha) = \dim_q(\beta) = q + 1/q$. An important part of the proof of Theorem 1.3 is based on the technical estimates provided by Lemma 6.1 and they require $q < 1$, i.e. $\dim_q(\alpha) > 2$.

Denote the q-factorials

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1} \; .$$

Writing $x = x_1 \otimes \cdots \otimes x_n$ where the words $x_1, \ldots, x_n$ are indecomposable, we have

$$\dim_q(x) = [|x_1| + 1]_q \cdots [|x_n| + 1]_q \; . \tag{3}$$

For later use, note that it follows that

$$\dim_q(xy) \geq q^{-|y|} \dim_q(x) \tag{4}$$

for all $x, y \in I$.

Whenever $x \in I \cup \partial I$, we denote by $[x]_n$ the word consisting of the first n letters of x and by $[x]^n$ the word that arises by removing the first n letters from x. So, by definition, $x = [x]_n[x]^n$.

2 Poisson boundary of the classical random walk on $\mathrm{Irred}(\mathbb{G})$

Given a probability measure μ on $I := \mathrm{Irred}(\mathbb{G})$, the Markov operator $P_\mu : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{G}})$ preserves the center $\mathcal{Z}(\ell^\infty(\widehat{\mathbb{G}})) = \ell^\infty(I)$ and hence, defines an ordinary random walk on the countable set I with n-step transition probabilities

$$p_x p_n(x, y) = p_x P_\mu^n(p_y) \; . \tag{5}$$

As shown above, this random walk is transient whenever $\mu(\epsilon) < 1$. Denote by $\mathrm{H}^\infty_{\mathrm{centr}}(\widehat{\mathbb{G}}, \mu)$ the commutative von Neumann algebra of bounded P_μ-harmonic functions in $\ell^\infty(I)$, with product given by $a \cdot b = \lim_n P_\mu^n(ab)$ and the sequence being strongly*-convergent. Write $p(x, y) = p_1(x, y)$.

The set I becomes in a natural way a tree, the Cayley tree of the semi-group $\mathbb{N} * \mathbb{N}$. Whenever μ is a generating probability measure with finite support, the following properties of the random walk on I can be checked easily.

- Uniform irreducibility: there exists an integer M such that, for any pair $x, y \in I$ of neighboring edges of the tree, there exists an integer $k \leq M$, such that $p_k(x, y) > 0$.
- Bounded step-length: there exists an integer N such that $p(x, y) > 0$ implies that $d(x, y) \leq N$ where $d(x, y)$ equals the length of the unique geodesic path from x to y.
- There exists a $\delta > 0$ such that $p(x, y) > 0$ implies that $p(x, y) \geq \delta$.

So, we can apply [9, Theorem 2] and identify the Poisson boundary of the random walk on I, with the boundary ∂I of infinite words in α,β, equipped with a probability measure in the following way.

Theorem 2.1 (Picardello and Woess, [9, Theorem 2])**.** *Let μ be a finitely supported generating measure on $I = \mathrm{Irred}(A_u(F))$, where F is not a scalar multiple of a 2×2 unitary matrix. Consider the associated random walk on I with transition probabilities given by (5) and the compactification $I \cup \partial I$ of I.*

- *The random walk converges almost surely to a point in ∂I.*
- *Denote, for every $x \in I$, by ν_x the hitting probability measure on ∂I, where $\nu_x(\mathcal{U})$ is defined as the probability that the random walk starting in x converges to a point in $\mathcal{U}$. Then, the formula*
$$\Upsilon(F)(x) = \int_{\partial I} F(z) d\nu_x(z) \tag{6}$$
defines a $*$-isomorphism $\Upsilon : \mathrm{L}^\infty(\partial I, \nu_\epsilon) \to \mathrm{H}^\infty_{\mathrm{centr}}(\widehat{\mathbb{G}}, \mu)$.

In fact, Theorem 2 in [9], identifies ∂I with the *Martin compactification* of the given random walk on I. It is a general fact (see [16, Theorem 24.10]), that a transient random walk converges almost surely to a point of the minimal Martin boundary and that the hitting probability measures provide a realization of the Poisson boundary through the Poisson integral formula (6), see [16, Theorem 24.12].

The rest of this section is devoted to the proof of the non-atomicity of the harmonic measures ν_x.

Lemma 2.2. *For all $x, y \in I$ and $z \in \partial I$, the sequence*

$$\Big(\frac{\dim_q(x[z]_n)}{\dim_q(y[z]_n)}\Big)_n$$

converges. By a slight abuse of notation, we denote the limit by $\dim_q\big(\frac{xz}{yz}\big)$. The map $\partial I \to \mathbb{R}_+ : z \mapsto \dim_q\big(\frac{xz}{yz}\big)$ is continuous.

Proof. Fix $x, y \in I$. Whenever $z \in \partial I$ and $n \in \mathbb{N}$, denote

$$f_n(z) = \frac{\dim_q(x[z]_n)}{\dim_q(y[z]_n)} \; .$$

If $z \notin \{\alpha\beta\alpha\cdots, \beta\alpha\beta\cdots\}$, write $z = z_1 \otimes z_2$ for some $z_1 \in I$, $z_1 \neq \epsilon$ and some $z_2 \in \partial I$. Denote by $\mathcal{U}$ the neighborhood of z consisting of words of the form $z_1 z' = z_1 \otimes z'$. For all $s \in \mathcal{U}$ and all $n \geq |z_1|$, we have

$$f_n(s) = \frac{\dim_q(xz_1)}{\dim_q(yz_1)} \; .$$

Hence, for all $s \in \mathcal{U}$, the sequence $n \mapsto f_n(s)$ is eventually constant and converges to a limit that is constant on $\mathcal{U}$.

Also for $z \in \{\alpha\beta\alpha\cdots, \beta\alpha\beta\cdots\}$, the sequence $f_n(z)$ is convergent. Take $z = \alpha\beta\alpha\cdots$. Write $x = x_0 \otimes x_1$ where x_1 is the longest possible (and maybe empty) indecomposable word ending with β. Write $y = y_0 \otimes y_1$ similarly. It follows that

$$f_n(z) = \frac{\dim_q(x_0)}{\dim_q(y_0)} \, \frac{[n + |x_1| + 1]_q}{[n + |y_1| + 1]_q} \to \frac{\dim_q(x_0)}{\dim_q(y_0)} \, q^{|y_1| - |x_1|} \; .$$

The convergence of $f_n(z)$ for $z = \beta\alpha\beta\cdots$ is proven analogously.

Write $f(z) = \lim_n f_n(z)$. It remains to prove that f is continuous in $z = \alpha\beta\alpha\cdots$ and in $z = \beta\alpha\beta\cdots$. In both cases, define for every $n \in \mathbb{N}$, the neighborhood $\mathcal{U}_n$ of z consisting of all $s \in \partial I$ with $[s]_n = [z]_n$. For every $s \in \mathcal{U}$, $s \neq z$, there exists $m \geq n$ such that $f(s) = f_m(z)$. The continuity of f in z follows. □

Whenever $x, y \in I \cup \partial I$, define $(x|y) := \max\{n \mid [x]_n = [y]_n\}$.

Lemma 2.3. *For all $x \in I$ and all $w \in \partial I$, we have*

$$\nu_x(w) = \frac{1}{\dim_q(x)} \sum_{k=0}^{(x|w)} \dim_q\left(\frac{[w]_k [w]^k}{\overline{[x]^k}[w]^k}\right) \nu_\epsilon\big(\overline{[x]^k}[w]^k\big) \; .$$

Proof. By Theorem 2.1, our random walk converges almost surely to a point of ∂I and we denoted by ν_x the hitting probability measure. So, $(\psi_x \otimes \psi_{\mu^{*n}})\hat{\Delta} \to \nu_x$ weakly* in $\mathrm{C}(I \cup \partial I)^*$.

Recall that $\mathcal{E} : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(I)$ denotes the conditional expectation defined by $\mathcal{E}(b)p_y = \psi_y(b)p_y$. Whenever $|z| > |x|$, we have

$$\mathcal{E}\big((\psi_x \otimes \mathrm{id})\hat{\Delta}(p_z)\big) = \sum_{k=0}^{(x|z)} \frac{\dim_q(z)}{\dim_q(x)\, \dim_q\big(\overline{[x]^k}[z]^k\big)} p_{\overline{[x]^k}[z]^k} \; .$$

Denote $q_z = \sum_{s \in I} p_{zs}$ and observe that $q_z \in \mathrm{C}(I \cup \partial I)$. It follows that for all $|z| > |x|$,

$$\mathcal{E}\big((\psi_x \otimes \mathrm{id})\hat{\Delta}(q_z)\big) = \sum_{k=0}^{(x|z)} f_k$$

where $f_k \in \ell^\infty(I)$ is defined by $f_k(y) = 0$ if y does not start with $\overline{[x]^k}[z]^k$ and

$$f_k(\overline{[x]^k}[z]^k y) = \frac{1}{\dim_q(x)} \frac{\dim_q(zy)}{\dim_q\big(\overline{[x]^k}[z]^k y\big)} \; .$$

By Lemma 2.2, we have $f_k \in \mathrm{C}(I \cup \partial I)$.

Defining $\mathcal{U}_z$ as the subset of ∂I consisting of infinite words starting with z, it follows that, for $|z| > |x|$

$$\nu_x(\mathcal{U}_z) = \sum_{k=0}^{(x|z)} \int_{\partial I} f_k(y) \; d\nu_\epsilon(y) \; . \tag{7}$$

The lemma follows by letting $z \to w$. ◻

Proposition 2.4. *The support of the harmonic measure ν_ϵ is the whole of ∂I. The harmonic measure ν_ϵ has no atoms in words ending with $\alpha\beta\alpha\beta\cdots$*

Remark 2.5. The same methods as in the proof of Proposition 2.4 given below, but involving more tedious computations, show in fact that ν_ϵ is non-atomic. To prove our main theorem, it is only crucial that ν_ϵ has no atoms in words ending with $\alpha\beta\alpha\beta\cdots$. We believe that it should be possible to give a more conceptual proof of the non-atomicity of ν_ϵ and refer to [15, Proposition 8.3.10] for an ad hoc proof along the lines of the proof of Proposition 2.4.

Proof of Proposition 2.4. Because of Lemma 2.3 and the equality

$$\nu_\epsilon = \sum_{x \in I} \mu^{*k}(x) \nu_x$$

for all $k \geq 1$, we observe that if w is an atom for ν_ϵ, then all w' with the same tail as w are atoms for all ν_x, $x \in I$. Denote by $\mathcal{U}_z \subset \partial I$ the open subset of ∂I consisting of infinite words that start with z. A similar argument, using (7), shows that if $\nu_\epsilon(\mathcal{U}_z) = 0$, then $\nu_x(\mathcal{U}_y) = 0$ for all $x, y \in I$, which is absurd because ν_x is a probability measure. So, the support of ν_ϵ is the whole of ∂I.

So, we assume now that $w := \alpha\beta\alpha\beta\cdots$ is an atom for ν_ϵ and derive a contradiction.

Denote by δ_w the function on ∂I that is equal to 1 in w and 0 elsewhere. Using the $*$-isomorphism in Theorem 2.1, it follows that the bounded function

$$\xi \in \ell^\infty(\widehat{\mathbb{G}}) : \xi(x) := \nu_x(w) = \int_{\partial I} \delta_w \; d\nu_x$$

is harmonic.

We will prove that ξ attains its maximum and apply the maximum principle for irreducible random walks (see e.g. [16, Theorem 1.15]) to deduce that ξ must be constant. This will lead to a contradiction.

Denote

$$w_n^\alpha := \underbrace{\alpha\beta\alpha\cdots}_{n \text{ letters}} \quad \text{and} \quad w_n^\beta := \underbrace{\beta\alpha\beta\cdots}_{n \text{ letters}} .$$

Note that all elements of I are either of the form

$$w_{2n+1}^\alpha x \quad \text{where } n \in \mathbb{N} \text{ and } x \in \{\epsilon\} \cup \alpha I$$

or of the form

$$w_{2n}^\alpha x \quad \text{where } n \in \mathbb{N} \text{ and } x \in \{\epsilon\} \cup \beta I .$$

By Lemma 2.3 and formula (3), we get that for $n \in \mathbb{N}$ and $x \in \{\epsilon\} \cup \alpha I$,

$$\begin{aligned}\xi(w_{2n+1}^\alpha x) &= \sum_{k=0}^{2n+1} \frac{1}{[2(n+1)]_q \ \dim_q(x)^2} \ \dim_q\Big(\frac{w_k^\alpha \qquad [w]^k}{w_{2n+1-k}^\beta \ [w]^k}\Big) \ \nu_\epsilon(\overline{x}\beta\alpha\beta\cdots) \\ &= \sum_{k=0}^{2n+1} \frac{1}{[2(n+1)]_q \ \dim_q(x)^2} \ q^{2(n-k)+1} \ \nu_\epsilon(\overline{x}\beta\alpha\beta\cdots) = \frac{\nu_\epsilon(\overline{x}\beta\alpha\beta\cdots)}{\dim_q(x)^2} .\end{aligned}$$

Since ν_ϵ is a probability measure, it follows that $x \mapsto \xi(w_{2n+1}^\alpha x)$ is independent of n and summable over the set $\{\epsilon\} \cup \alpha I$. Analogously, it follows that $x \mapsto \xi(w_{2n}^\alpha x)$ is independent of n and summable over the set $\{\epsilon\} \cup \beta I$. As a result, ξ attains its maximum on I. By the maximum principle, ξ is constant. Since $\xi(\epsilon) \neq 0$, this constant is non-zero and we arrive at a contradiction with the summability of $x \mapsto \xi(w_{2n+1}^\alpha x)$ over the infinite set $\{\epsilon\} \cup \alpha I$. $\square$

3 Topological boundary and boundary action for the dual of $A_u(F)$

Before proving Theorem 1.3, we construct a compactification for $\widehat{\mathbb{G}}$, i.e. a unital C*-algebra $\mathcal{B}$ lying between $c_0(\widehat{\mathbb{G}})$ and $\ell^\infty(\widehat{\mathbb{G}})$. This C*-algebra $\mathcal{B}$ is a non-commutative version of $C(I \cup \partial I)$. The construction of $\mathcal{B}$ follows word by word the analogous construction given in [11, Section 3] for $\mathbb{G} = A_o(F)$. So, we only indicate the necessary modifications.

For all $x, y \in I$ and $z \subset x \otimes y$, we choose an isometry $V(x \otimes y, z) \in \mathrm{Mor}(x \otimes y, z)$. Since z appears with multiplicity one in $x \otimes y$, the isometry $V(x \otimes y, z)$ is uniquely determined up to multiplication by a scalar $\lambda \in S^1$. Therefore, the following unital completely positive maps are uniquely defined (cf. [11, Definition 3.1]).

Definition 3.1. Let $x, y \in I$. We define unital completely positive maps

$$\psi_{xy,x} : \mathcal{L}(H_x) \to \mathcal{L}(H_{xy}) : \psi_{xy,x}(A) = V(x \otimes y, xy)^*(A \otimes 1)V(x \otimes y, xy) .$$

Theorem 3.2. *The maps $\psi_{xy,x}$ form an inductive system of completely positive maps. Defining*

$$\begin{aligned}\mathcal{B} = \{a \in \ell^\infty(\widehat{\mathbb{G}}) \ | \quad &\forall \varepsilon > 0, \ \exists n \in \mathbb{N} \quad \text{such that} \quad \|ap_{xy} - \psi_{xy,x}(ap_x)\| < \varepsilon \\ &\text{for all} \quad x, y \in I \quad \text{with} \quad |x| \geq n\} ,\end{aligned}$$

we get that $\mathcal{B}$ is a unital C-subalgebra of $\ell^\infty(\widehat{\mathbb{G}})$ containing $c_0(\widehat{\mathbb{G}})$.*

- *The restriction of the comultiplication $\hat{\Delta}$ yields a left action $\beta_{\widehat{\mathbb{G}}}$ of $\widehat{\mathbb{G}}$ on $\mathcal{B}$:*

$$\beta_{\widehat{\mathbb{G}}} : \mathcal{B} \to \mathrm{M}(c_0(\widehat{\mathbb{G}}) \otimes \mathcal{B}) : a \mapsto \hat{\Delta}(a) .$$

- *The restriction of the adjoint action of* $\mathbb{G}$ *on* $\ell^\infty(\widehat{\mathbb{G}})$ *yields a right action of* $\mathbb{G}$ *on* $\mathcal{B}$ *:*

$$\beta_{\mathbb{G}} : \mathcal{B} \to \mathcal{B} \otimes \mathrm{C}(\mathbb{G}) : a \mapsto \mathbb{V}(a \otimes 1)\mathbb{V}^* \; .$$

Here, $\mathbb{V} \in \ell^\infty(\widehat{\mathbb{G}})\overline{\otimes}\mathrm{L}^\infty(\mathbb{G})$ *is defined as* $\mathbb{V} = \sum_{x\in I} U^x$*. The action* $\beta_{\mathbb{G}}$ *is continuous in the sense that* $\operatorname{span}\beta_{\mathbb{G}}(\mathcal{B})(1 \otimes \mathrm{C}(\mathbb{G}))$ *is dense in* $\mathcal{B} \otimes \mathrm{C}(\mathbb{G})$*.*

Proof. One can repeat word by word the proofs of [11, Propositions 3.4 and 3.6]. The crucial ingredients of these proofs are the approximate commutation formulae provided by [11, Lemmas A.1 and A.2] and they have to be replaced by the inequalities provided by Lemma 6.1. □

We denote $\mathcal{B}_\infty := \mathcal{B}/\,\mathrm{c}_0(\widehat{\mathbb{G}})$ and call it the topological boundary of $\widehat{\mathbb{G}}$. Both actions $\beta_{\mathbb{G}}$ and $\beta_{\widehat{\mathbb{G}}}$ preserve the ideal $\mathrm{c}_0(\widehat{\mathbb{G}})$ and hence yield actions on $\mathcal{B}_\infty$ that we still denote by $\beta_{\mathbb{G}}$ and $\beta_{\widehat{\mathbb{G}}}$.

We identify $\mathrm{C}(I \cup \partial I)$ with $\mathcal{B} \cap \mathcal{Z}(\ell^\infty(\widehat{\mathbb{G}})) = \mathcal{B} \cap \ell^\infty(I)$. Similarly, $\mathrm{C}(\partial I) \subset \mathcal{B}_\infty$.

We partially order I by writing $x \leq y$ if $y = xz$ for some $z \in I$. Define

$$\psi_{\infty,x} : \mathcal{L}(H_x) \to \mathcal{B} : \psi_{\infty,x}(A)p_y = \begin{cases} \psi_{y,x}(A) & \text{if } \; y \geq x \\ 0 & \text{else} \end{cases} \; .$$

We use the same notation for the composition of $\psi_{\infty,x}$ with the quotient map $\mathcal{B} \to \mathcal{B}_\infty$, yielding the map $\psi_{\infty,x} : \mathcal{L}(H_x) \to \mathcal{B}_\infty$.

Lemma 3.3. *The inclusion* $\mathrm{C}(\partial I) \subset \mathcal{B}_\infty$ *defines a continuous field of unital C*-algebras. Denote, for every* $x \in \partial I$*, by* J_x *the closed two-sided ideal of* $\mathcal{B}_\infty$ *generated by the functions in* $\mathrm{C}(\partial I)$ *vanishing in* x*.*

For every $x = x_1 \otimes x_2 \otimes \cdots$ *in* $\partial_0 I$*, there exists a unique surjective* $*$*-homomorphism*

$$\pi_x : \mathcal{B}_\infty \to \bigotimes_{k=1}^{\infty} \mathcal{L}(H_{x_k})$$

satisfying $\operatorname{Ker}\pi_x = J_x$ *and* $\pi_x(\psi_{\infty,x_1\cdots x_n}(A)) = A \otimes 1$ *for all* $A \in \bigotimes_{k=1}^n \mathcal{L}(H_{x_k}) = \mathcal{L}(H_{x_1\cdots x_n})$*.*

Proof. Given $x \in \partial I$, define the decreasing sequence of projections $e_n \in \mathcal{B}$ given by

$$e_n := \sum_{y\in I} p_{[x]_n y} \; .$$

Denote by $\pi : \mathcal{B} \to \mathcal{B}_\infty$ the quotient map. It follows that

$$\|\pi(a) + J_x\| = \lim_n \|a e_n\| \tag{8}$$

for all $a \in \mathcal{B}$.

To prove that $\mathrm{C}(\partial I) \subset \mathcal{B}_\infty$ is a continuous field, let $y \in I$, $A \in \mathcal{L}(H_y)$ and define $a \in \mathcal{B}$ by $a := \psi_{\infty,y}(A)$. Put $f : \partial I \to \mathbb{R}_+ : f(x) = \|\pi(a) + J_x\|$. We have to prove that f is a continuous function. Define $\mathcal{U} \subset \partial I$ consisting of infinite words starting with y. Then, $\mathcal{U}$ is open and closed and f is zero, in particular continuous, on the complement of $\mathcal{U}$. Assume that the last letter of y is α (the other case, of course, being analogous). If $x \in \mathcal{U}$ and $x \neq y\beta\alpha\beta\alpha\cdots$, write $x = yz \otimes u$ for some $z \in I$, $u \in \partial I$. Define $\mathcal{V}$ as the neighborhood of x consisting of infinite words of the form

yzu' where $u' \in \partial I$ and $yzu' = yz \otimes u'$. Then, f is constantly equal to $\|\psi_{yz,y}(A)\|$ on $\mathcal{V}$. It remains to prove that f is continuous in $x := y\beta\alpha\beta\alpha\cdots$. Let

$$w_n = \underbrace{\beta\alpha\beta\cdots}_{n \text{ letters}} .$$

Then, the sequence $\|\psi_{yw_n,y}(A)\|$ is decreasing and converges to $f(x)$. If $\mathcal{U}_n$ is the neighborhood of x consisting of words starting with yw_n, it follows that

$$f(x) \leq f(u) \leq \|\psi_{yw_n,y}(A)\|$$

for all $u \in \mathcal{U}_n$. This proves the continuity of f in x. So, $\mathrm{C}(\partial I) \subset \mathcal{B}_\infty$ is a continuous field of C*-algebras.

Let now $x = x_1 \otimes x_2 \otimes \cdots$ be an element of $\partial_0 I$. Put $y_n = x_1 \otimes \cdots \otimes x_n$ and

$$f_n := \sum_{z \in I} p_{y_n z} .$$

The map $A \mapsto f_{n+1}\psi_{\infty,y_n}(A)$ defines a unital $*$-homomorphism from $\mathcal{L}(H_{y_n})$ to $f_{n+1}\mathcal{B}$. Since $1 - f_{n+1} \in J_x$, we obtain the unital $*$-homomorphism $\theta_n : \mathcal{L}(H_{y_n}) \to \mathcal{B}/J_x$. The $*$-homomorphisms θ_n are compatible and combine into the unital $*$-homomorphism

$$\theta : \bigotimes_{k=1}^{\infty} \mathcal{L}(H_{x_k}) \to \mathcal{B}/J_x .$$

By (8), θ is isometric. Since the union of all $\psi_{\infty,y_n}(\mathcal{L}(H_{y_n})) + J_x$, $n \in \mathbb{N}$, is dense in $\mathcal{B}$, it follows that θ is surjective. The composition of the quotient map $\mathcal{B} \to \mathcal{B}/J_x$ and the inverse of θ provides the required $*$-homomorphism π_x. $\square$

4 Proof of Theorem 1.3

We prove Theorem 1.3 by performing the following steps.

- Construct on the boundary $\mathcal{B}_\infty$ of $\widehat{\mathbb{G}}$, a faithful KMS state ω_∞, to be considered as the harmonic state and satisfying $(\psi_\mu \otimes \omega_\infty)\beta_{\widehat{\mathbb{G}}} = \omega_\infty$. Extend $\beta_{\widehat{\mathbb{G}}}$ to an action

$$\beta_{\widehat{\mathbb{G}}} : (\mathcal{B}_\infty, \omega_\infty)'' \to \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} (\mathcal{B}_\infty, \omega_\infty)''$$

  and denote by $\Theta_\mu := (\mathrm{id} \otimes \omega_\infty)\beta_{\widehat{\mathbb{G}}}$ the *Poisson integral.*

- Prove a quantum *Dirichlet property :* for all $a \in \mathcal{B}$, we have $\Theta_\mu(a) - a \in \mathrm{c}_0(\widehat{\mathbb{G}})$. It will follow that Θ_μ is a normal and faithful $*$-homomorphism of $(\mathcal{B}_\infty, \omega_\infty)''$ onto a von Neumann subalgebra of $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$.

- By Theorem 2.1, Θ_μ is a $*$-isomorphism of $\mathrm{L}^\infty(\partial I, \nu_\epsilon) \subset (\mathcal{B}_\infty, \omega_\infty)''$ onto $\mathrm{H}^\infty_{\mathrm{centr}}(\widehat{\mathbb{G}}, \mu)$. Deduce that the image of Θ_μ is the whole of $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$.

- Use Lemma 3.3 to identify $(\mathcal{B}_\infty, \omega_\infty)''$ with a field of ITPFI factors.

Proposition 4.1. *The sequence $\psi_{\mu^{*n}}$ of states on $\mathcal{B}$ converges weakly* to a KMS state ω_∞ on $\mathcal{B}$. The state ω_∞ vanishes on $\mathrm{c}_0(\widehat{\mathbb{G}})$. We still denote by ω_∞ the resulting KMS state on $\mathcal{B}_\infty$. Then, ω_∞ is faithful on $\mathcal{B}_\infty$.*

We have $(\psi_\mu \otimes \omega_\infty)\beta_{\widehat{\mathbb{G}}} = \omega_\infty$, so that we can uniquely extend $\beta_{\widehat{\mathbb{G}}}$ to an action

$$\beta_{\widehat{\mathbb{G}}} : (\mathcal{B}_\infty, \omega_\infty)'' \to \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} (\mathcal{B}_\infty, \omega_\infty)''$$

that we still denote by $\beta_{\widehat{\mathbb{G}}}$.

The state ω_∞ is invariant under the action $\beta_{\mathbb{G}}$ of $\mathbb{G}$ on $\mathcal{B}_\infty$. We extend $\beta_{\mathbb{G}}$ to an action on $(\mathcal{B}_\infty, \omega_\infty)''$ that we still denote by $\beta_{\mathbb{G}}$.

The normal, completely positive map

$$\Theta_\mu : (\mathcal{B}_\infty, \omega_\infty)'' \to \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu) : \Theta_\mu = (\mathrm{id} \otimes \omega_\infty)\beta_{\widehat{\mathbb{G}}} \tag{9}$$

is called the Poisson integral. *It satisfies the following properties (recall that $\alpha_{\widehat{\mathbb{G}}}$ and $\alpha_{\mathbb{G}}$ were introduced in Definition 1.2).*

- $\widehat{\epsilon} \circ \Theta_\mu = \omega_\infty$.
- $(\Theta_\mu \otimes \mathrm{id}) \circ \beta_{\mathbb{G}} = \alpha_{\mathbb{G}} \circ \Theta_\mu$.
- $(\mathrm{id} \otimes \Theta_\mu) \circ \beta_{\widehat{\mathbb{G}}} = \alpha_{\widehat{\mathbb{G}}} \circ \Theta_\mu$.

For every $x = x_1 \otimes x_2 \otimes \cdots$ in $\partial_0 I$, denote by ω_x the infinite tensor product state on $\bigotimes_{k=1}^\infty \mathcal{L}(H_{x_k})$, of the states ψ_{x_k} on $\mathcal{L}(H_{x_k})$. Using the notation π_x of Lemma 3.3, we have

$$\omega_\infty(a) = \int_{\partial_0 I} \omega_x(\pi_x(a))\, d\nu_\epsilon(x) \tag{10}$$

for all $a \in \mathcal{B}_\infty$.

Proof. Define the 1-parameter group of automorphisms $(\sigma_t)_{t\in\mathbb{R}}$ of $\ell^\infty(\widehat{\mathbb{G}})$ given by

$$\sigma_t(a)p_x = Q_x^{it} a p_x Q_x^{-it} \; .$$

Since $\sigma_t(\psi_{\infty,x}(A)) = \psi_{\infty,x}(Q_x^{it} A Q_x^{-it})$, it follows that (σ_t) is norm-continuous on the C*-algebra $\mathcal{B}$.

By Theorem 2.1, the sequence of probability measures μ^{*n} on $I \cup \partial I$ converges weakly* to ν_ϵ. It follows that $\psi_{\mu^{*n}}(a) \to 0$ whenever $a \in \mathrm{c}_0(\widehat{\mathbb{G}})$. Given $x \in I$ and $A \in \mathcal{L}(H_x)$, put $a := \psi_{\infty,x}(A)$. Denote by $\partial(xI)$ the set of infinite words starting with x and by $\partial_0(xI)$ its intersection with $\partial_0(I)$. We get, using Proposition 2.4,

$$\begin{aligned} \psi_{\mu^{*n}}(a) &= \sum_{y\in I} \mu^{*n}(y)\psi_y(\psi_{\infty,x}(A)) = \sum_{y \in xI} \mu^{*n}(y)\psi_x(A) \\ &\to \psi_x(A)\nu_\epsilon(\partial(xI)) = \psi_x(A)\nu_\epsilon(\partial_0(xI)) = \int_{\partial_0 I} \omega_y(\pi_y(a))\, d\nu_\epsilon(y) \; . \end{aligned}$$

So, the sequence $\psi_{\mu^{*n}}$ of states on $\mathcal{B}$ converges weakly* to a state on $\mathcal{B}$ that we denote by ω_∞ and that satisfies (10). Since all $\psi_{\mu^{*n}}$ satisfy the KMS condition w.r.t. (σ_t), also ω_∞ is a KMS state. If $a \in \mathcal{B}_\infty^+$ and $\omega_\infty(a) = 0$, it follows from (10) that $\omega_x(\pi_x(a)) = 0$ for ν_ϵ-almost every $x \in \partial_0 I$. Since

ω_x is faithful, it follows that $\|\pi(a)+J_x\|=0$ for ν_ϵ-almost every $x\in\partial I$. By Proposition 2.4, the support of ν_ϵ is the whole of ∂I and by Lemma 3.3, $x\mapsto\|\pi(a)+J_x\|$ is a continuous function. It follows that $\|\pi(a)+J_x\|=0$ for all $x\in\partial I$ and hence, $a=0$. So, ω_∞ is faithful.

Since $(\psi_\mu\otimes\psi_{\mu^{*n}})\beta_{\widehat{\mathbb{G}}}=\psi_{\mu^{*(n+1)}}$, it follows that $(\psi_\mu\otimes\omega_\infty)\beta_{\widehat{\mathbb{G}}}=\omega_\infty$. So, $(\psi_{\mu^{*k}}\otimes\omega_\infty)\beta_{\widehat{\mathbb{G}}}=\omega_\infty$ for all $k\in\mathbb{N}$. Since μ is generating, there exists for every $x\in I$, a $C_x>0$ such that $(\psi_x\otimes\omega_\infty)\beta_{\widehat{\mathbb{G}}}\leq C_x\omega_\infty$. As a result, we can uniquely extend $\beta_{\widehat{\mathbb{G}}}$ to a normal $*$-homomorphism

$$(\mathcal{B}_\infty,\omega_\infty)''\to\ell^\infty(\widehat{\mathbb{G}})\overline{\otimes}(\mathcal{B}_\infty,\omega_\infty)''\ .$$

Since $\beta_{\widehat{\mathbb{G}}}$ is an action, the same holds for the extension to the von Neumann algebra $(\mathcal{B}_\infty,\omega_\infty)''$.

Because $(\psi_\mu\otimes\omega_\infty)\beta_{\widehat{\mathbb{G}}}=\beta_{\widehat{\mathbb{G}}}$ and because $\beta_{\widehat{\mathbb{G}}}$ is an action, the Poisson integral defined by (9) takes values in $\mathrm{H}^\infty(\widehat{\mathbb{G}},\mu)$. It is straightforward to check that Θ_μ intertwines $\beta_{\mathbb{G}}$ with $\alpha_{\mathbb{G}}$ and $\beta_{\widehat{\mathbb{G}}}$ with $\alpha_{\widehat{\mathbb{G}}}$. □

Theorem 4.2. *The compactification $\mathcal{B}$ of $\widehat{\mathbb{G}}$ satisfies the quantum Dirichlet property, meaning that, for all $a\in\mathcal{B}$,*

$$\|(\Theta_\mu(a)-a)p_x\|\to 0$$

if $|x|\to\infty$.

In particular, the Poisson integral Θ_μ is a normal and faithful $*$-homomorphism of $(\mathcal{B}_\infty,\omega_\infty)''$ onto a von Neumann subalgebra of $\mathrm{H}^\infty(\widehat{\mathbb{G}},\mu)$.

We will deduce Theorem 4.2 from the following lemma.

Lemma 4.3. *For every $a\in\mathcal{B}$ we have that*

$$\sup_{y\in I}\|(\mathrm{id}\otimes\psi_y)\hat{\Delta}(a)p_x-ap_x\|\to 0 \tag{11}$$

when $|x|\to\infty$.

Proof. Fix $a\in\mathcal{B}$ with $\|a\|\leq 1$. Choose $\varepsilon>0$. Take n such that $\|ap_{x_0x_1}-\psi_{x_0x_1,x_0}(ap_{x_0})\|<\varepsilon$ for all $x_0,x_1\in I$ with $|x_0|=n$.

Denote $d_{S^1}(V,W)=\inf\{\|V-\lambda W\|\mid\lambda\in S^1\}$. By (16) below, take k such that

$$\begin{aligned}d_{S^1}\big(&(V(x_0\otimes x_1x_2,x_0x_1x_2)\otimes 1)V(x_0x_1x_2\otimes\overline{x_2}u,x_0x_1u)\ ,\\&(1\otimes V(x_1x_2\otimes\overline{x_2}u,x_1u))V(x_0\otimes x_1u,x_0x_1u)\big)<\frac{\varepsilon}{2}\end{aligned} \tag{12}$$

whenever $|x_1|\geq k$.

Finally, take l such that $q^{2l}<\varepsilon$. We prove that

$$\|(\mathrm{id}\otimes\psi_y)\hat{\Delta}(a)p_x-ap_x\|<5\varepsilon \tag{13}$$

for all $x,y\in I$ with $|x|\geq n+k+l$.

Choose $x, y \in I$ with $|x| \geq n+k+l$ and write $x = x_0 x_1 x_2$ with $|x_0| = n$, $|x_1| = k$ and hence, $|x_2| \geq l$. We get

$$
\begin{aligned}
(\mathrm{id} \otimes \psi_y)\hat{\Delta}(a)p_x &= \sum_{z \subset x \otimes y} (\mathrm{id} \otimes \psi_y)\big(V(x \otimes y, z) a p_z V(x \otimes y, z)^*\big) \\
&= \sum_{z \subset x \otimes y} \frac{\dim_q(z)}{\dim_q(x)\ \dim_q(y)} V(z \otimes \overline{y}, x)^* (a p_z \otimes 1) V(z \otimes \overline{y}, x) \\
&= \sum_{z \subset x_2 \otimes y} \frac{\dim_q(x_0 x_1 z)}{\dim_q(x)\ \dim_q(y)} V(x_0 x_1 z \otimes \overline{y}, x)^* (a p_{x_0 x_1 z} \otimes 1) V(x_0 x_1 z \otimes \overline{y}, x) \\
&\qquad + \sum \text{remaining terms} .
\end{aligned}
$$

In order to have remaining terms, y should be of the form $y = \overline{x_2} y_0$ and then, using (4) and the assumption $\|a\| \leq 1$,

$$
\begin{aligned}
\sum \|\text{remaining terms}\| &= \sum_{z \subset x_0 x_1 \otimes y_0} \frac{\dim_q(z)}{\dim_q(x_0 x_1 x_2)\ \dim_q(\overline{x_2} y_0)} \\
&\leq \sum_{z \subset x_0 x_1 \otimes y_0} q^{2|x_2|} \frac{\dim_q(z)}{\dim_q(x_0 x_1)\ \dim_q(y_0)} = q^{2|x_2|} < \varepsilon .
\end{aligned}
$$

Combining this estimate with the fact that $\|a p_{x_0 x_1 z} - \psi_{x_0 x_1 z, x_0}(a p_{x_0})\| < \varepsilon$, it follows that

$$
\begin{aligned}
&\big\|(\mathrm{id} \otimes \psi_y)\hat{\Delta}(a)p_x - a p_x\big\| \\
&\quad \leq 2\varepsilon + \Big\| a p_x - \sum_{z \subset x_2 \otimes y} \frac{\dim_q(x_0 x_1 z)}{\dim_q(x)\ \dim_q(y)} V(x_0 x_1 z \otimes \overline{y}, x)^* (\psi_{x_0 x_1 z, x_0}(a p_{x_0}) \otimes 1) V(x_0 x_1 z \otimes \overline{y}, x) \Big\| .
\end{aligned}
$$

But now, (12) implies that

$$
\big\|(\mathrm{id} \otimes \psi_y)\hat{\Delta}(a)p_x - a p_x\big\| \leq 3\varepsilon + \Big\| a p_x - \sum_{z \subset x_2 \otimes y} \frac{\dim_q(x_0 x_1 z)}{\dim_q(x)\ \dim_q(y)} \psi_{x, x_0}(a p_{x_0}) \Big\| .
$$

Since $\|\psi_{x,x_0}(a p_{x_0}) - a p_x\| < \varepsilon$ and $\|a\| \leq 1$, we get

$$
\big\|(\mathrm{id} \otimes \psi_y)\hat{\Delta}(a)p_x\big\| \leq 4\varepsilon + \Big(1 - \sum_{z \subset x_2 \otimes y} \frac{\dim_q(x_0 x_1 z)}{\dim_q(x)\ \dim_q(y)}\Big) .
$$

The second term on the right hand side is zero, unless $y = \overline{x_2} y_0$, in which case, it equals

$$
\sum_{z \subset x_0 x_1 \otimes y_0} \frac{\dim_q(z)}{\dim_q(x_0 x_1 x_2)\ \dim_q(\overline{x_2} y_0)} \leq \sum_{z \subset x_0 x_1 \otimes y_0} q^{2|x_2|} \frac{\dim_q(z)}{\dim_q(x_0 x_1)\ \dim_q(y_0)} \leq \varepsilon
$$

because of (4). Finally, (13) follows and the lemma is proven. □

Proof of Theorem 4.2. Let $a \in \mathcal{B}$. Given $\varepsilon > 0$, Lemma 4.3 provides k such that

$$
\|(\mathrm{id} \otimes \psi_{\mu^{*n}})\hat{\Delta}(a)p_x - a p_x\| \leq \varepsilon
$$

for all $n \in \mathbb{N}$ and all x with $|x| \geq k$. Since $\psi_{\mu^{*n}} \to \omega_\infty$ weakly*, it follows that

$$
\|(\Theta_\mu(a) - a)p_x\| \leq \epsilon
$$

whenever $|x| \geq k$. This proves (11).

It remains to prove the multiplicativity of Θ_μ. We know that $\Theta_\mu : \mathcal{B}_\infty \to \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$ is a unital, completely positive map. Since $\widehat{\epsilon} \circ \Theta_\mu = \omega_\infty$, Θ_μ is faithful. Denote by $\pi : \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu) \to \frac{\ell^\infty(\widehat{\mathbb{G}})}{c_0(\widehat{\mathbb{G}})}$ the quotient map, which is also a unital, completely positive map. By (11), we have $\pi \circ \Theta_\mu = \mathrm{id}$. So, for all $a \in \mathcal{B}_\infty$, we find

$$\pi(\Theta_\mu(a)^* \cdot \Theta_\mu(a)) \leq \pi(\Theta_\mu(a^*a)) = a^*a = \pi(\Theta_\mu(a))^*\pi(\Theta_\mu(a)) \leq \pi(\Theta_\mu(a)^* \cdot \Theta_\mu(a))$$

We claim that π is faithful. If $a \in \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)^+ \cap c_0(\widehat{\mathbb{G}})$, we have $\widehat{\epsilon}(a) = \psi_{\mu^{*n}}(a)$ for all n and the transience of μ combined with the assumption $a \in c_0(\widehat{\mathbb{G}})$, implies that $\widehat{\epsilon}(a) = 0$ and hence, $a = 0$. So, we conclude that $\Theta_\mu(a)^* \cdot \Theta_\mu(a) = \Theta_\mu(a^*a)$ for all $a \in \mathcal{B}_\infty$. Hence, Θ_μ is multiplicative on $\mathcal{B}_\infty$ and also on $(\mathcal{B}_\infty, \omega_\infty)''$ by normality. □

Remark 4.4. We now give a reinterpretation of Theorem 2.1. Denote by $\Omega = I^{\mathbb{N}}$ the path space of the random walk with transition probabilities (5). Elements of Ω are denoted by $\underline{x}$, $\underline{y}$, etc. For every $x \in I$, one defines the probability measure $\mathbb{P}_x$ on Ω such that $\mathbb{P}_x(\{x\} \times I \times I \times \cdots) = 1$ and

$$\mathbb{P}_x(\{(x, x_1, x_2, \cdots, x_n)\} \times I \times I \times \cdots) = p(x, x_1)\, p(x_1, x_2) \;\cdots\; p(x_{n-1}, x_n)\;.$$

Choose a probability measure η on I with $I = \operatorname{supp} \eta$. Write $\mathbb{P} = \sum_{x \in I} \eta(x) \mathbb{P}_x$.

Define on Ω the following equivalence relation: $\underline{x} \sim \underline{y}$ iff there exist $k, l \in \mathbb{N}$ such that $x_{n+k} = y_{n+l}$ for all $n \in \mathbb{N}$. Whenever $F \in \mathrm{H}^\infty_{\mathrm{centr}}(\widehat{\mathbb{G}}, \mu)$, the martingale convergence theorem implies that the sequence of measurable functions $\Omega \to \mathbb{C} : \underline{x} \mapsto F(x_n)$ converges $\mathbb{P}$-almost everywhere to a $\sim$-invariant bounded measurable function on Ω, that we denote by $\pi_\infty(F)$. Denote by $\mathrm{L}^\infty(\Omega/_\sim, \mathbb{P})$ the von Neumann subalgebra of $\sim$-invariant functions in $\mathrm{L}^\infty(\Omega, \mathbb{P})$. As such, $\pi_\infty : \mathrm{H}^\infty_{\mathrm{centr}}(\widehat{\mathbb{G}}, \mu) \to \mathrm{L}^\infty(\Omega/_\sim, \mathbb{P})$ is a $*$-isomorphism.

By Theorem 4.2, we can define the measurable function $\mathrm{bnd} : \Omega \to \partial I$ such that $\mathrm{bnd}\, \underline{x} = \lim_n x_n$ for $\mathbb{P}$-almost every $\underline{x} \in \Omega$ and where the convergence is understood in the compact space $I \cup \partial I$. Recall that we denote, for $x \in I$, by ν_x the hitting probability measure on ∂I. So, $\nu_x(A) = \mathbb{P}_x(\mathrm{bnd}^{-1}(A))$ for all measurable $A \subset \partial I$ and all $x \in I$.

Again by Theorem 4.2, $\pi_\infty \circ \Upsilon$ is a $*$-isomorphism of $\mathrm{L}^\infty(\partial I, \nu_\epsilon)$ onto $\mathrm{L}^\infty(\Omega/_\sim, \mathbb{P})$. We claim that for all $F \in \mathrm{L}^\infty(\partial I, \nu_\epsilon)$, we have

$$\big((\pi_\infty \circ \Upsilon)(F)\big)(\underline{x}) = F(\mathrm{bnd}\, \underline{x}) \quad \text{for } \mathbb{P}\text{-almost every } \; \underline{x} \in \Omega\;.$$

Let $A \subset \partial I$ be measurable. Define $F_n : \Omega \to \mathbb{R} : F_n(\underline{x}) = \nu_{x_n}(A)$. Then, F_n converges almost everywhere with limit equal to $(\pi_\infty \circ \Upsilon)(\chi_A)$. If the measurable function $G : \Omega \to \mathbb{C}$ only depends on $x_0, \ldots, x_k$, one checks that

$$\int_\Omega F_n(\underline{x})\; G(\underline{x})\; d\mathbb{P}(\underline{x}) = \int_{\mathrm{bnd}^{-1}(A)} G(\underline{x})\; d\mathbb{P}(\underline{x}) \quad \text{for all } \; n > k\;.$$

From this, the claim follows.

Since the $*$-isomorphism $\pi_\infty \circ \Upsilon$ is given by bnd, it follows that for every $\sim$-invariant bounded measurable function F on Ω, there exists a bounded measurable function F_1 on ∂I such that $F(\underline{x}) = F_1(\mathrm{bnd}\, \underline{x})$ for $\mathbb{P}$-almost every path $\underline{x} \in \Omega$.

We are now ready to prove the main Theorem 1.3.

Proof of Theorem 1.3. Because of Theorem 4.2 and Lemma 3.3, it remains to show that

$$\Theta_\mu : (\mathcal{B}_\infty, \omega_\infty)'' \to \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$$

is surjective.

Whenever $\gamma : N \to N \overline{\otimes} \mathrm{L}^\infty(\mathbb{G})$ is an action of $\mathbb{G}$ on the von Neumann algebra N, we denote, for $x \in I$, by $N^x \subset N$ the *spectral subspace* of the irreducible representation x. By definition, N^x is the linear span of all $S(H_x)$, where S ranges over the linear maps $S : H_x \to N$ satisfying $\gamma(S(\xi)) = (S \otimes \mathrm{id})(U_x(\xi \otimes 1))$. The linear span of all N^x, $x \in I$, is a weakly dense $*$-subalgebra of N, called the spectral subalgebra of N. For $n \in \mathbb{N}$, we denote by N^n the linear span of all N^x, $|x| \leq n$.

Fixing $x, y \in I$, consider the adjoint action $\gamma : \mathcal{L}(H_{xy}) \to \mathcal{L}(H_{xy}) \otimes \mathrm{C}(\mathbb{G})$ given by $\gamma(A) = U_{xy}(A \otimes 1)U_{xy}^*$. The fusion rules of $\mathbb{G} = A_u(F)$ imply that $\mathcal{L}(H_{xy})^{2|x|} = \psi_{xy,x}(\mathcal{L}(H_x))$.

For the rest of the proof, put $M := (\mathcal{B}_\infty, \omega_\infty)''$. We use the action $\beta_\mathbb{G}$ of $\mathbb{G}$ on M and the action $\alpha_\mathbb{G}$ of $\mathbb{G}$ on $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$. Fix $k \in \mathbb{N}$. It suffices to prove that $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)^k \subset \Theta_\mu(M)$.

Define, for all $y \in I$, the subset

$$V_y := \{yz \mid z \in I \text{ and } yz = y \otimes z\} .$$

Define the projections

$$q_y = \sum_{z \in V_y} p_z \in \mathcal{B}$$

and consider q_y also as an element of the von Neumann algebra M. Define $W_y \subset \partial I$ as the subset of infinite words of the form yu, where $u \in \partial I$ and $yu = y \otimes u$.

Define the unital $*$-homomorphism

$$\zeta : \mathrm{C}(W_y) \otimes \mathcal{L}(H_y) \to Mq_y : \begin{cases} \zeta(F)p_{yz} = \psi_{yz,y}(F(yz)) & \text{when } y \otimes z = yz \\ 0 & \text{else} \end{cases} .$$

Claim. For all $y \in I$, there exists a linear map

$$T_y : \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)^{2|y|} \cdot \Theta_\mu(q_y) \subset \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu) \to \mathrm{L}^\infty(W_y) \otimes \mathcal{L}(H_y)$$

satisfying the following conditions.

- T_y is isometric for the 2-norm on $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$ given by the state $\widehat{\epsilon}$ and the 2-norm on $\mathrm{L}^\infty(W_y) \otimes \mathcal{L}(H_y)^k$ given by the state $\nu_\varepsilon \otimes \psi_y$.
- $(T_y \circ \Theta_\mu \circ \zeta)(F) = F$ for all $F \in \mathrm{C}(W_y) \otimes \mathcal{L}(H_y)$.

To prove this claim, we use the notations and results introduced in Remark 4.4. Fix $y \in I$. Consider $a \in \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)^{2|y|} \cdot \Theta_\mu(q_y)$. If $\underline{x} \in \Omega$ is such that $\mathrm{bnd}(\underline{x}) \in W_y$, then, for n big enough, x_n will be of the form $x_n = y \otimes z_n$. By definition of $\alpha_\mathbb{G}$, we have that $ap_{x_n} \in \mathcal{L}(H_{x_n})^{2|y|}$. So, we can take elements $a_{\underline{x},n} \in \mathcal{L}(H_y)$ such that $ap_{x_n} = \psi_{x_n,y}(a_{\underline{x},n})$. We prove that, for $\mathbb{P}$-almost every path $\underline{x}$ with $\mathrm{bnd}\,\underline{x} \in W_y$, the sequence $(a_{\underline{x},n})_n$ is convergent. We then define $T_y(a) \in \mathrm{L}^\infty(W_y) \otimes \mathcal{L}(H_y)$ such that $T_y(a)(\mathrm{bnd}\,\underline{x}) = \lim_n a_{\underline{x},n}$ for $\mathbb{P}$-almost every path $\underline{x}$ with $\mathrm{bnd}\,\underline{x} \in W_y$.

Take $d \in \mathcal{L}(H_y)$. Then, for $\mathbb{P}$-almost every path $\underline{x}$ such that $\operatorname{bnd} \underline{x} \in W_y$ and n big enough, we get that

$$\psi_y(da_{\underline{x},n}) = \psi_{x_n}\big(\psi_{x_n,y}(da_{\underline{x},n})\big) = \psi_{x_n}\big(\psi_{x_n,y}(d)\psi_{x_n,y}(a_{\underline{x},n})\big) = \psi_{x_n}\big(\psi_{x_n,y}(d)ap_{x_n}\big)$$

In the second step, we used the multiplicativity of $\psi_{x_n,y} : \mathcal{L}(H_y) \to \mathcal{L}(H_{x_n})$ which follows because $x_n = y \otimes z_n$. Also note that $\|a_{\underline{x},n}\| \leq \|a\|$. From Theorem 4.2, it follows that

$$\big\|\Theta_\mu\big(\zeta(1 \otimes d)\big)p_{x_n} - \psi_{x_n,y}(d)p_{x_n}\big\| \to 0$$

whenever x_n converges to a point in W_y. This implies that

$$\big|\psi_y(da_{\underline{x},n}) - \psi_{x_n}\big(\Theta_\mu\big(\zeta(1 \otimes d)\big)ap_{x_n}\big)\big| \to 0$$

for $\mathbb{P}$-almost every path $\underline{x}$ with $\operatorname{bnd} \underline{x} \in W_y$.

From [6, Proposition 3.3], we know that for $\mathbb{P}$-almost every path $\underline{x}$,

$$\big|\psi_{x_n}\big(\Theta_\mu(\zeta(1 \otimes d))ap_{x_n}\big)p_{x_n} - \mathcal{E}\big(\Theta_\mu(\zeta(1 \otimes d)) \cdot a\big)p_{x_n}\big| \to 0 \; .$$

As before, $\mathcal{E}(b)p_x = \psi_x(b)p_x$. It follows that

$$\big|\psi_y(da_{\underline{x},n})p_{x_n} - \mathcal{E}\big(\Theta_\mu\big(\zeta(1 \otimes d)\big) \cdot a\big)p_{x_n}\big| \to 0 \; .$$

Note that $\mathcal{E}$ maps $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$ onto $\mathrm{H}^\infty_{\text{centr}}(\widehat{\mathbb{G}}, \mu)$. Whenever $F \in \mathrm{H}^\infty_{\text{centr}}(\widehat{\mathbb{G}}, \mu)$, the sequence $F(x_n)$ converges for $\mathbb{P}$-almost every path $\underline{x}$. We conclude that for every $d \in \mathcal{L}(H_y)$, the sequence $\psi_y(da_{\underline{x},n})$ is convergent for $\mathbb{P}$-almost every path $\underline{x}$ with $\operatorname{bnd} \underline{x} \in W_y$. Since $\mathcal{L}(H_y)$ is finite dimensional, it follows that the sequence $(a_{\underline{x},n})_n$ in $\mathcal{L}(H_y)$ is convergent for $\mathbb{P}$-almost every path $\underline{x}$ with $\operatorname{bnd} \underline{x} \in W_y$.

By Remark 4.4, we get $T_y(a) \in \mathrm{L}^\infty(W_y) \otimes \mathcal{L}(H_y)$ such that $T_y(a)(\operatorname{bnd} \underline{x}) = \lim_n a_{\underline{x},n}$ for $\mathbb{P}$-almost every path $\underline{x}$ with $\operatorname{bnd} \underline{x} \in W_y$. From the definition of $a_{\underline{x},n}$, we get that

$$\big\|\psi_{x_n,y}\big(T_y(a)(\operatorname{bnd} \underline{x})\big) - ap_{x_n}\big\| \to 0 \tag{14}$$

for $\mathbb{P}$-almost every path $\underline{x}$ such that $\operatorname{bnd} \underline{x} \in W_y$.

The map T_y is isometric. Indeed, by the defining property (14) and again by [6, Proposition 3.3], we have, for $\mathbb{P}$-almost every path $\underline{x}$ with $\operatorname{bnd} \underline{x} \in W_y$,

$$\psi_y\big(T_y(a)(\operatorname{bnd} \underline{x})^* T_y(a)(\operatorname{bnd} \underline{x})\big) = \lim_{n\to\infty} \psi_{x_n}(a^* a p_{x_n}) = (\pi_\infty \circ \mathcal{E})(a^* \cdot a)(\underline{x}) \; .$$

Here, π_∞ denotes the $*$-isomorphism $\mathrm{H}^\infty_{\text{centr}}(\widehat{\mathbb{G}}, \mu) \to \mathrm{L}^\infty(\Omega/_\sim, \mathbb{P})$ introduced in Remark 4.4. On the other hand, by Remark 4.4, $\big((\pi_\infty \circ \Theta_\mu)(q_y)\big)(\underline{x}) = 0$ for $\mathbb{P}$-almost every path $\underline{x}$ with $\operatorname{bnd} \underline{x} \notin W_y$. Since

$$\int_\Omega \big((\pi_\infty \circ \mathcal{E})(b)\big)(\underline{x}) \; d\mathbb{P}_\epsilon(\underline{x}) = \widehat{\epsilon}(b)$$

for all $b \in \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)$, it follows that T_y is an isometry in 2-norm.

We next prove that $(T_y \circ \Theta_\mu \circ \zeta)(F) = F$ for all $F \in \mathrm{C}(W_y) \otimes \mathcal{L}(H_y)$. Let $\widetilde{a} \in \mathrm{C}(I \cup \partial I) \subset \ell^\infty(I)$ and let a be the restriction of $\widetilde{a}$ to ∂I. Take $A \in \mathcal{L}(H_y)$. It suffices to take $F = a \otimes A$. Theorem 4.2 implies that

$$\big\|\widetilde{a} p_{x_n} \psi_{x_n,y}(A) - (\Theta_\mu \circ \zeta)(a \otimes A)p_{x_n}\big\| \to 0$$

for $\mathbb{P}$-almost every path $\underline{x}$. On the other hand, for $\mathbb{P}$-almost every path $\underline{x}$ with $\operatorname{bnd}\underline{x} \in W_y$, the scalar $\widetilde{a}p_{x_n}$ converges to $a(\operatorname{bnd}\underline{x})$. In combination with (14), it follows that $(T_y \circ \Theta_\mu \circ \zeta)(a \otimes A) = a \otimes A$, concluding the proof of the claim.

Having proven the claim, we now show that for all $y \in I$, $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)^{2|y|} \cdot \Theta_\mu(q_y) \subset \Theta_\mu(M)$. Take $a \in \mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)^{2|y|} \cdot \Theta_\mu(q_y)$. Let d_n be a bounded sequence in the C*-algebra $\mathrm{C}(W_y) \otimes \mathcal{L}(H_y)$ converging to $T_y(a)$ in 2-norm. Since $T_y \circ \Theta_\mu$ is an isometry in 2-norm, it follows that $\zeta(d_n)$ is a bounded sequence in M that converges in 2-norm. Denoting by $c \in M$ the limit of $\zeta(d_n)$, we conclude that $T_y(\Theta_\mu(c)) = T_y(a)$ and hence, $\Theta_\mu(c) = a$.

Fix $k \in \mathbb{N}$. A fortiori, $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)^k \cdot \Theta_\mu(q_y) \subset \Theta_\mu(M)$ for all $y \in I$ with $2|y| \geq k$. By Proposition 2.4, the harmonic measure ν_ϵ has no atoms in infinite words ending with $\alpha\beta\alpha\beta\cdots$. As a result, 1 is the smallest projection in M that dominates all q_y, $y \in I$, $2|y| \geq k$. So, $\mathrm{H}^\infty(\widehat{\mathbb{G}}, \mu)^k \subset \Theta_\mu(M)$ for all $k \in \mathbb{N}$. This finally implies that Θ_μ is surjective. $\square$

5 Solidity and the Akemann-Ostrand property

In Section 3, we followed the approach of [11] to construct the compactification $\mathcal{B}$ of $\widehat{\mathbb{G}}$. In fact, more of the constructions and results of [11] carry over immediately to the case $\mathbb{G} = A_u(F)$. We continue to assume that F is not a multiple of a 2×2 unitary matrix.

Denote by $\mathrm{L}^2(\mathbb{G})$ the GNS Hilbert space defined by the Haar state h on $\mathrm{C}(\mathbb{G})$. Denote by $\lambda : \mathrm{C}(\mathbb{G}) \to \mathcal{L}(\mathrm{L}^2(\mathbb{G}))$ the corresponding GNS representation and define $C_{\mathrm{red}}(\mathbb{G}) := \lambda(\mathrm{C}(\mathbb{G}))$. We can view λ as the left-regular representation. We also have a right-regular representation ρ and the operators $\lambda(a)$ and $\rho(b)$ commute for all $a, b \in \mathrm{C}(\mathbb{G})$ (see [11, Formulae (1.3)]).

Repeating the proofs of [11, Proposition 3.8 and Theorem 4.5], we arrive at the following result.

Theorem 5.1. *The boundary action $\beta_{\widehat{\mathbb{G}}}$ of $\widehat{\mathbb{G}}$ on $\mathcal{B}$ defined in Theorem 3.2 is*

- amenable *in the sense of [11, Definition 4.1];*
- small at infinity *in the following sense: the comultiplication $\hat{\Delta}$ restricts as well to a right action of $\widehat{\mathbb{G}}$ on $\mathcal{B}$; this action leaves $\mathrm{c}_0(\widehat{\mathbb{G}})$ globally invariant and becomes the trivial action on the quotient $\mathcal{B}_\infty$.*

By construction, $\mathcal{B}$ is a nuclear C*-algebra and hence, as in [11, Corollary 4.7], we get that

- $\mathbb{G}$ satisfies the Akemann-Ostrand property: the homomorphism
$$C_{\mathrm{red}}(\mathbb{G}) \otimes_{\mathrm{alg}} C_{\mathrm{red}}(\mathbb{G}) \to \frac{\mathcal{L}(\mathrm{L}^2(\mathbb{G}))}{\mathcal{K}(\mathrm{L}^2(\mathbb{G}))} : a \otimes b \mapsto \lambda(a)\rho(b) + \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$$
is continuous for the minimal C*-tensor product $\otimes_{\min}$.
- $C_{\mathrm{red}}(\mathbb{G})$ is an exact C*-algebra.

As before, we denote by $\mathrm{L}^\infty(\mathbb{G})$ the von Neumann algebra acting on $\mathrm{L}^2(\mathbb{G})$ generated by $\lambda(\mathrm{C}(\mathbb{G}))$. From [2, Théorème 3], it follows that $\mathrm{L}^\infty(\mathbb{G})$ is a factor, of type II_1 if F is a multiple of an $n \times n$ unitary matrix and of type III in the other cases.

Applying [8, Theorem 6] (in fact, its slight generalization provided by [11, Theorem 2.5]), we get the following corollary of Theorem 5.1. Recall that a II_1 factor M is called *solid* if for every diffuse von Neumann subalgebra $A \subset M$, the relative commutant $M \cap A'$ is injective. An arbitrary von Neumann algebra M is called *generalized solid* if the same holds for every diffuse von Neumann subalgebra $A \subset M$ which is the image of a faithful normal conditional expectation.

Corollary 5.2. *When $n \geq 3$ and $\mathbb{G} = A_u(I_n)$, the II_1 factor $\mathrm{L}^\infty(\mathbb{G})$ is solid. When $n \geq 2$, $F \in \mathrm{GL}(n, \mathbb{C})$ is not a multiple of an $n \times n$ unitary matrix and $\mathbb{G} = A_u(F)$, the type III factor $\mathrm{L}^\infty(\mathbb{G})$ is generalized solid.*

6 Appendix: approximate intertwining relations

We fix an invertible matrix F and assume that F is not a scalar multiple of a unitary 2×2 matrix. Define $\mathbb{G} = A_u(F)$ and label the irreducible representations of $\mathbb{G}$ by the monoid $\mathbb{N} * \mathbb{N}$, freely generated by α and β. The representation labeled by α is the fundamental representation of $\mathbb{G}$ and β is its contragredient. Define $0 < q < 1$ such that $\dim_q(\alpha) = \dim_q(\beta) = q + \frac{1}{q}$. Recall from Section 3 that whenever $z \subset x \otimes y$, we choose an isometry $V(x \otimes y, z) \in \mathrm{Mor}(x \otimes y, z)$. Observe that $V(x \otimes y, z)$ is uniquely determined up to multiplication by a scalar $\lambda \in S^1$. We denote by $p_z^{x \otimes y}$ the projection $V(x \otimes y, z)V(x \otimes y, z)^*$.

Lemma 6.1. *There exists a constant $C > 0$ that only depends on q such that*

$$\begin{aligned}
&\|\big(V(xr \otimes \overline{r}y, xy) \otimes 1_z\big)p_{xyz}^{xy \otimes z} - (1_{xr} \otimes p_{\overline{r}yz}^{\overline{r}y \otimes z})\big(V(xr \otimes \overline{r}y, xy) \otimes 1_z\big)\| \leq Cq^{|y|} \; , \\
&\|\big(1_x \otimes V(yr \otimes \overline{r}z, yz)\big)p_{xyz}^{x \otimes yz} - (p_{xyr}^{x \otimes yr} \otimes 1_{\overline{r}z})\big(1_x \otimes V(yr \otimes \overline{r}z, yz)\big)\| \leq Cq^{|y|}
\end{aligned} \tag{15}$$

for all $x, y, z, r \in I$.

One way of proving Lemma 6.1 consists in repeating step by step the proof of [11, Lemma A.1]. But, as we explain now, Lemma 6.1 can also be deduced more directly from [11, Lemma A.1].

Sketch of proof. Whenever $y = y_1 \otimes y_2$ with $y_1 \neq \epsilon \neq y_2$, the expressions above are easily seen to be 0. Denote

$$v_n = \underbrace{\alpha \otimes \beta \otimes \alpha \otimes \cdots}_{n \text{ tensor factors}} \quad \text{and} \quad w_n = \underbrace{\beta \otimes \alpha \otimes \beta \otimes \cdots}_{n \text{ tensor factors}} \; .$$

The remaining estimates that have to be proven, reduce to estimates of norms of operators in $\mathrm{Mor}(v_n, v_m)$ and $\mathrm{Mor}(w_n, w_m)$. Putting these spaces together in an infinite matrix, one defines the C*-algebras

$$A := \big(\mathrm{Mor}(v_n, v_m)\big)_{n,m} \quad \text{and} \quad B := \big(\mathrm{Mor}(w_n, w_m)\big)_{n,m}$$

generated by the subspaces $\mathrm{Mor}(v_n, v_m)$ and $\mathrm{Mor}(w_n, w_m)$, respectively. Choose unit vectors $t \in \mathrm{Mor}(\alpha \otimes \beta, \epsilon)$ and $s \in \mathrm{Mor}(\beta \otimes \alpha, \epsilon)$ such that $(t^* \otimes 1)(1 \otimes s) = (q + 1/q)^{-1}$. By [2, Lemme 5], the C*-algebra A is generated by the elements $1^{\otimes 2k} \otimes t \otimes 1^{\otimes l}$, $1^{\otimes (2k+1)} \otimes s \otimes 1^{\otimes l}$. A similar statement holds for B.

Denote by U the fundamental representation of the quantum group $\mathrm{SU}_{-q}(2)$ and let $t_0 \in \mathrm{Mor}(U \otimes U, \epsilon)$ be a unit vector. The proofs of [3, Theorems 5.3 and 6.2] (which heavily rely on results in [1, 2]), imply the existence of $*$-isomorphisms

$$\pi_A : (\mathrm{Mor}(U^{\otimes n}, U^{\otimes m}))_{n,m} \to A \quad \text{and} \quad \pi_B : (\mathrm{Mor}(U^{\otimes n}, U^{\otimes m}))_{n,m} \to B$$

satisfying

$$\pi_A(1^{\otimes 2k} \otimes t_0 \otimes 1^{\otimes l}) = 1^{\otimes 2k} \otimes t \otimes 1^{\otimes l} \quad \text{and} \quad \pi_A(1^{\otimes(2k+1)} \otimes t_0 \otimes 1^{\otimes l}) = 1^{\otimes(2k+1)} \otimes s \otimes 1^{\otimes l}$$

and similarly for π_B.

As a result, the estimates to be proven, follow directly from the corresponding estimates for $\mathrm{SU}_{-q}(2)$ proven in [11, Lemma A.1]. $\square$

Using the notation

$$d_{S^1}(V, W) = \inf\{\|V - \lambda W\| \mid \lambda \in S^1\} ,$$

several approximate commutation relations can be deduced from Lemma 6.1. For instance, after a possible increase of the constant C, (15) implies that

$$d_{S^1}\Big(\big(1_x \otimes V(yr \otimes \overline{r}z, yz)\big)V(x \otimes yz, xyz) \; , \; \big(V(x \otimes yr, xyr) \otimes 1_{\overline{r}z}\big)V(xyr \otimes \overline{r}z, xyz)\Big) \leq Cq^{|y|} \tag{16}$$

for all $x, y, z, r \in I$. We again refer to [11, Lemma A.1] for a full list of approximate intertwining relations.

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